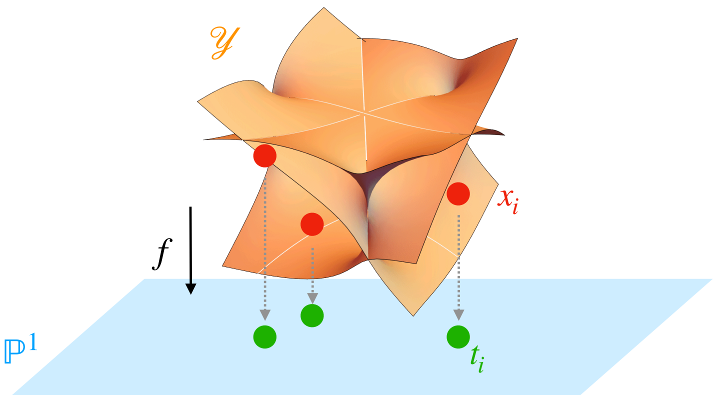
Eric Pichon-Pharabod

# Numerical Computation of the Homology and Periods of Complex Surfaces

Joint work with Pierre Lairez and Pierre Vanhove



# Periods are integrals of rational fractions

$A$ is homogeneous of degree $k \deg P - \deg \Omega$

$\Omega$ is the volume form of $\mathbb{P}^n$

$$\int_\gamma \frac{A}{P^k}\Omega$$

Some integration domain without boundary

$P$ defines a smooth complex projective hypersurface $\mathscr{X} = V(P) = \{P = 0\}$

# The period matrix

We chose generating families $\gamma_1, \ldots, \gamma_r \in H_n(\mathscr{X})$ and $\omega_1, \ldots, \omega_r \in H^n_{DR}(\mathscr{X})$.

Define the period matrix

$$\Pi = \left( \int_{\gamma_j} \omega_i \right)_{\substack{1 \leq i \leq r \\ 1 \leq j \leq r}}$$

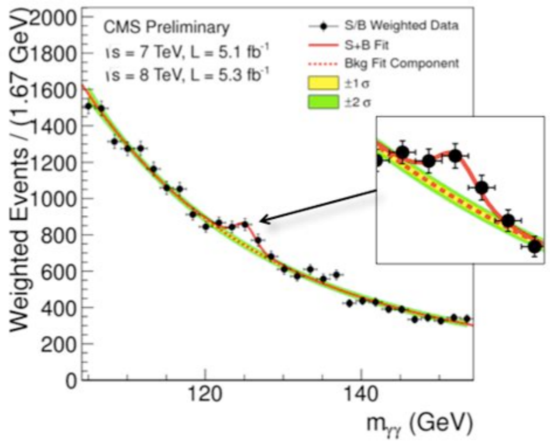
It is an **invertible** matrix that describes the isomorphism between DeRham cohomology and homology.

Our goal is to find a way, given $P$, to compute the period matrix of $\mathscr{X} = V(P)$.

# Why are periods interesting?

The period matrix of $\mathscr{X}$ contains information about fine **algebraic invariants** $\mathscr{X}$.
**Torelli-type theorems** : the period matrix of $\mathscr{X}$ determines its isomorphism class (in certain cases).

**Feynman integrals** are relative periods that give scattering amplitudes of particle interactions in quantum field theory.



# Previous works

**[Deconinck, van Hoeij 2001], [Bruin, Sijsling, Zotine 2018], [Molin, Neurohr 2017]:**
Algebraic curves (Riemann surfaces)

**[Elsenhans, Jahnel 2018], [Cynk, van Straten 2019]:**
Higher dimensional varieties (double covers of $\mathbb{P}^2$ ramified along 6 lines / of $\mathbb{P}^3$ ramified along 8 planes)

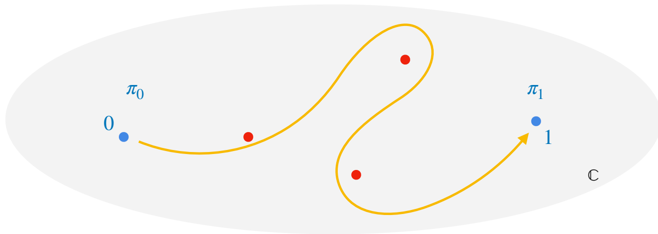
**[Sertöz 2019]:** compute the period matrix by **deformation.**

# Previous works

**Sertöz 2019:** compute the periods matrix by **deformation :**

We wish to compute $\displaystyle\int_\gamma \frac{\Omega}{X^3+Y^3+Z^3+XYZ}$.

Let us consider instead $\displaystyle\pi_t = \int_{\gamma_t} \frac{\Omega}{X^3+Y^3+Z^3+tXYZ}$,




Exact formulae are known for $\pi_0$ **[Pham 65, Sertöz 19]**

Furthermore $\pi_t$ is a solution to the differential operator
$\mathscr{L} = (t^3+27)\partial_t^2 + 3t^2\partial_t + t$ (Picard-Fuchs equation)

We may numerically compute the analytic continuation of $\pi_0$
along a path from 0 to 1 **[Chudnovsky², Van der Hoeven, Mezzarobba]**
This way, we obtain a numerical approximation of $\pi_1$.

# Previous works

**Sertöz 2019:** compute the periods matrix by **deformation :**

Two drawbacks :

We rely on the knowledge of the periods of some variety.
**[Pham 65, Sertöz 19]** provides the periods of the Fermat hypersurfaces $V(X_0^d + \ldots + X_n^d)$.
In more general cases (e.g. complete intersections), we do not have this data.

The differential operators that need to be integrated quickly go beyond what current software can manage:
To compute the periods of a smooth quartic surface in $\mathbb{P}^3$,
one needs to integrate an operator of order 21.

**Goal:** a more intrinsic description of the integrals should solve both problems.

# Contributions

Hundreds of digits ↙

New method for computing periods with high precision:

→ implementation in Sagemath (relying on OreAlgebra) — lefschetz_family

→ sufficiently efficient to compute periods of new varieties (generic quartic surface)

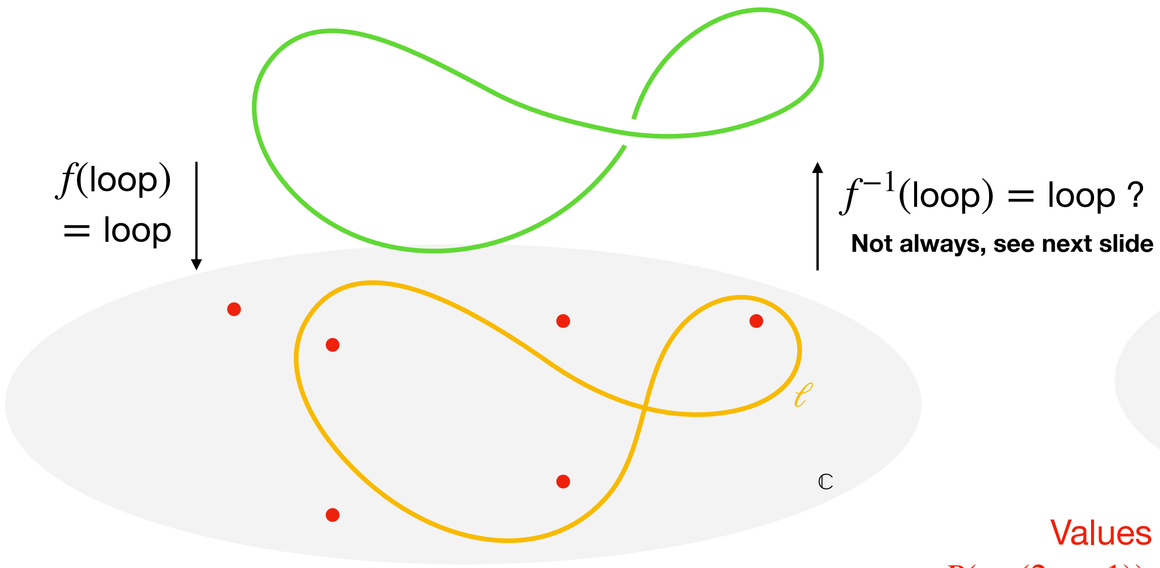
→ homology of complex algebraic varieties

→ generalisable to other types of varieties (e.g. complete intersections, varieties with isolated singularities, etc.)

# First example: algebraic curves

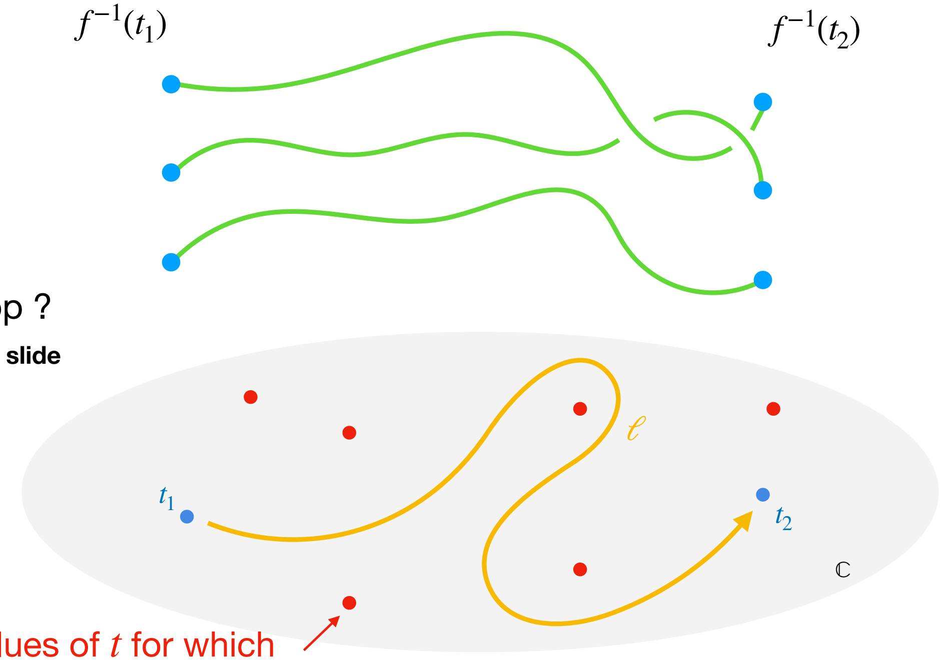
Let $\mathscr{X}$ be the elliptic curve defined by $P = y^3 + x^3 + 1 = 0$ and let $f : (x, y) \mapsto y/(2x+1)$.

In dimension 1, we are looking for closed paths in $\mathscr{X}$, up to deformation (1-cycles).



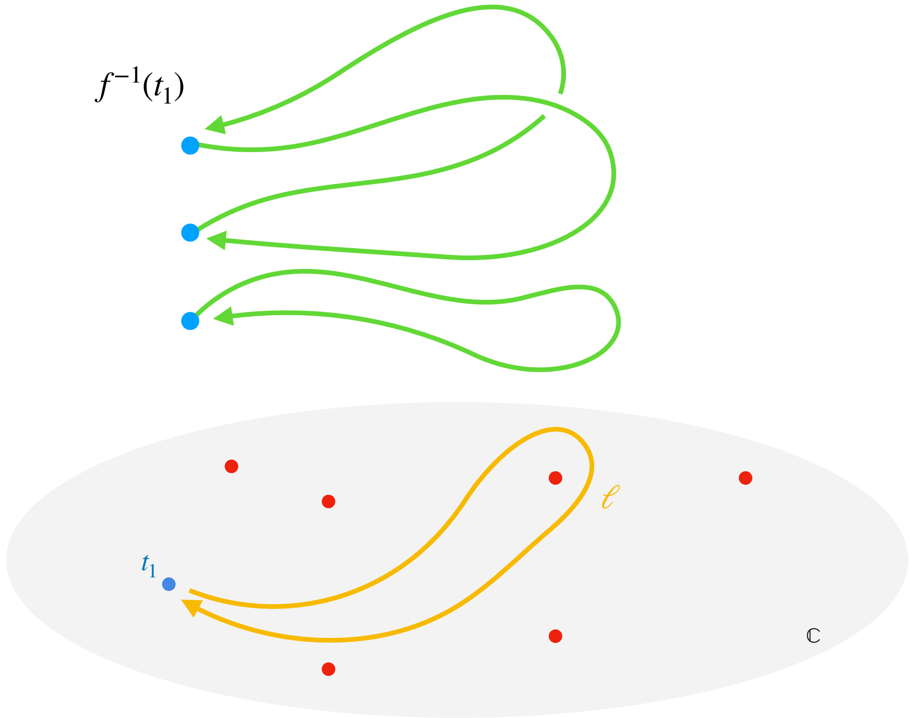

The fibre above $t \in \mathbb{C}$ is $\mathscr{X}_t = f^{-1}(t)$
$= \{(x, t(2x+1)) \mid P\left(x, t(2x+1)\right) = 0\}$.

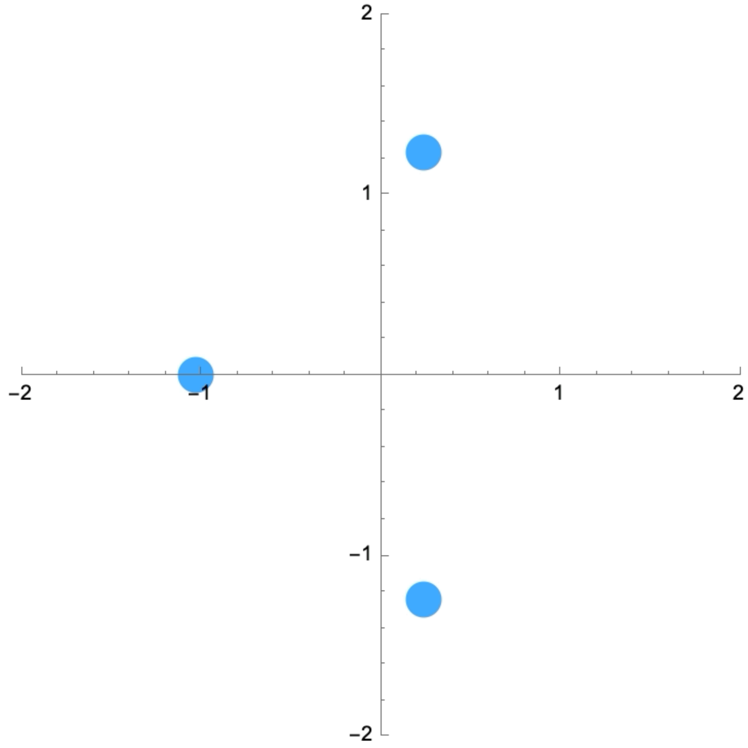
It deforms continuously with respect to $t$.



# What happens when you loop around a critical point?

A loop $\ell$ in $\mathbb{C}$ pointed at $t_1$ induces a permutation of $\mathcal{X}_{t_1} = f^{-1}(t_1)$.



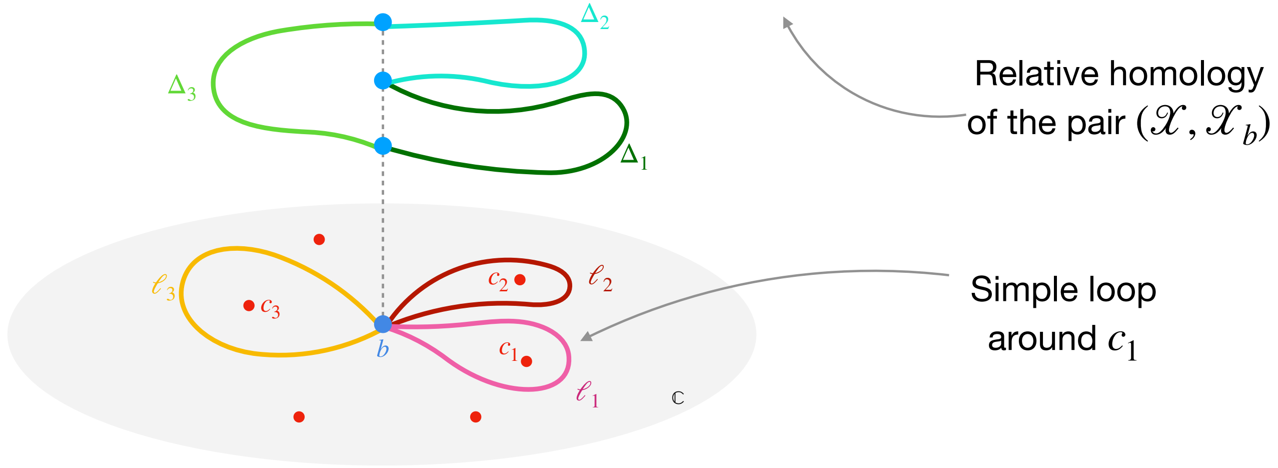




This permutation is called the **action of monodromy along** $\ell$ on $\mathcal{X}_{t_1}$. It is denoted $\ell_*$

If $\ell$ is a simple loop around a critical value, $\ell_*$ is a transposition.

# Periods of algebraic curves

The lift of a simple loop $\ell$ around a critical value $c$ that has a non-trivial boundary in $\mathcal{X}_b$ is called the **thimble** of $c$. It is an element of $H_1(\mathcal{X}, \mathcal{X}_b)$.




Thimbles serve as building blocks to recover $H_1(\mathcal{X})$.
It is sufficient to glue thimbles together in a way such that their boundaries cancels.

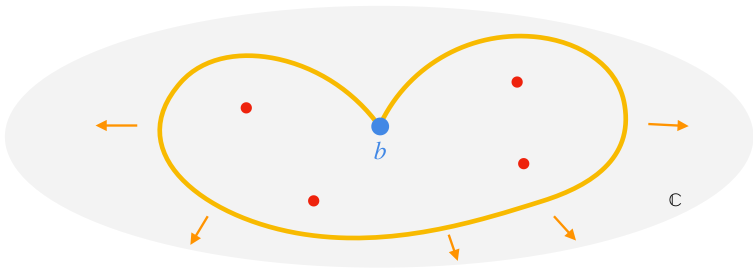
Concretely, we take the kernel of the boundary map $\delta : H_1(\mathcal{X}, \mathcal{X}_b) \to H_0(\mathcal{X}_b)$

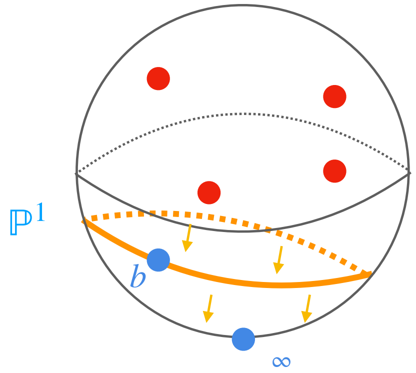
**Fact:** all of $H_1(\mathcal{X})$ can be recovered this way.

$$0 \to H_1(\mathcal{X}) \to H_1(\mathcal{X}, \mathcal{X}_b) \to H_0(\mathcal{X}_b)$$

Generated by thimbles

# Certain combinations of thimbles are trivial



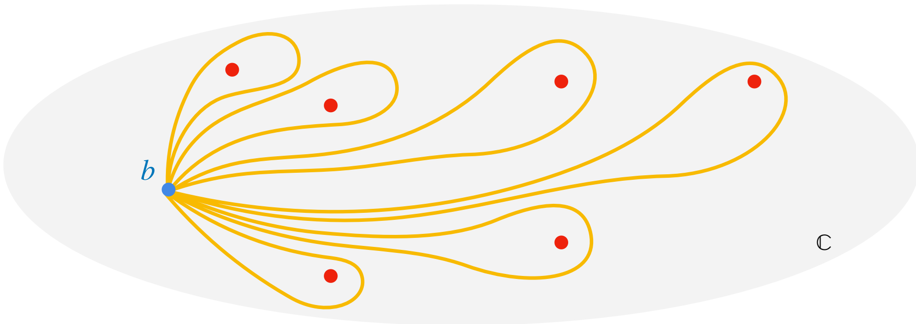




Extensions along contractible paths in $\mathbb{P}^1 \setminus \{\text{crit. val.}\}$ have a trivial homology class in $H_1(\mathscr{X})$.

**Fact:** these are the only ones — the kernel of the map $\mathbb{Z}^r \mapsto H_1(\mathscr{X}, \mathscr{X}_b)$, $k_1, \ldots, k_r \mapsto \sum_i k_i \Delta_i$ is generated by these extensions "around infinity".

# Computing periods of algebraic curves

1. Compute simple loops $\ell_1, \ldots, \ell_{\#\text{crit.}}$ around the critical values — basis of $\pi_1(\mathbb{C} \setminus \{\text{crit. val.}\})$



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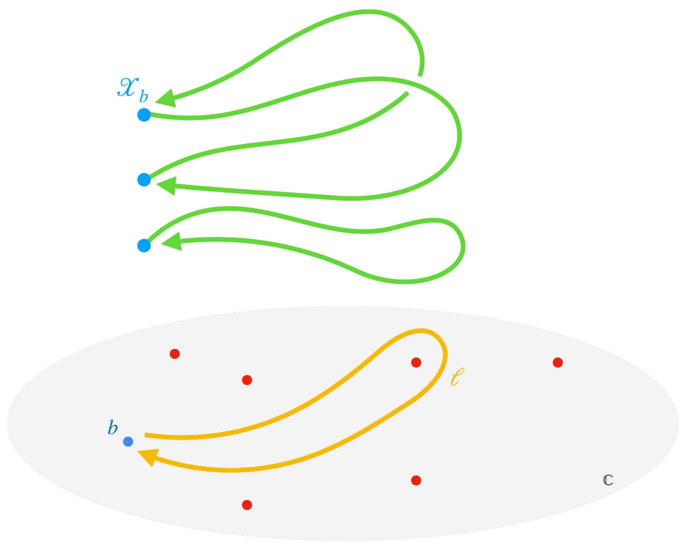

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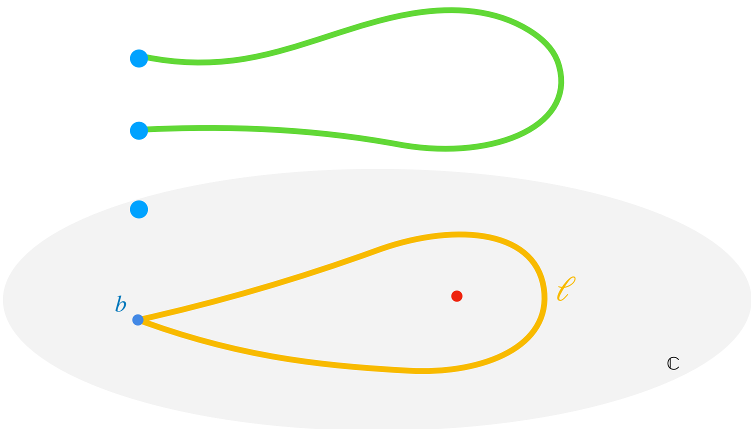
3. This provides the thimble $\Delta_i$. Its boundary is the difference of the two points of $\mathcal{X}_b$ that are permuted.



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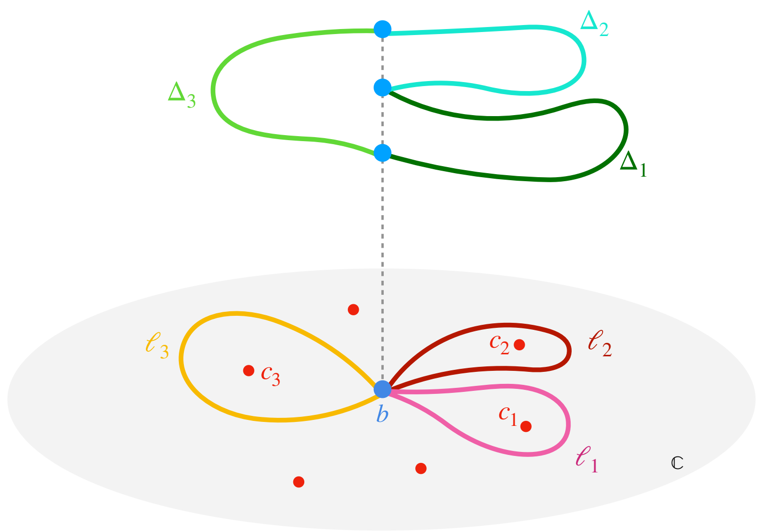
4. Compute sums of thimbles without boundary → basis of $H_1(\mathscr{X})$



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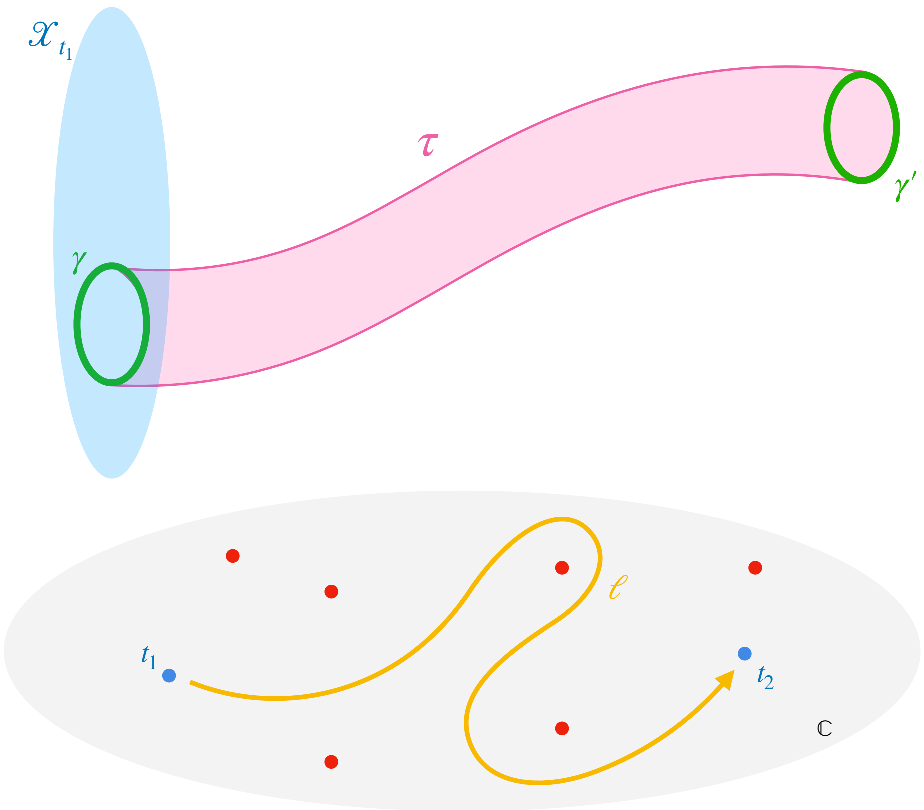
5. Periods are integrals along these loops
→ we have an explicit parametrisation of these paths → numerical integration.

$$\int_\gamma \omega = \int_\ell \omega_t$$

**DEMO**

# Insight into higher dimensions: surfaces

We take a projection $\mathscr{X} \to \mathbb{P}^1$.
The fibre $\mathscr{X}_t$ is a variety of dimension 1.
It deforms continuously with respect to $t$.



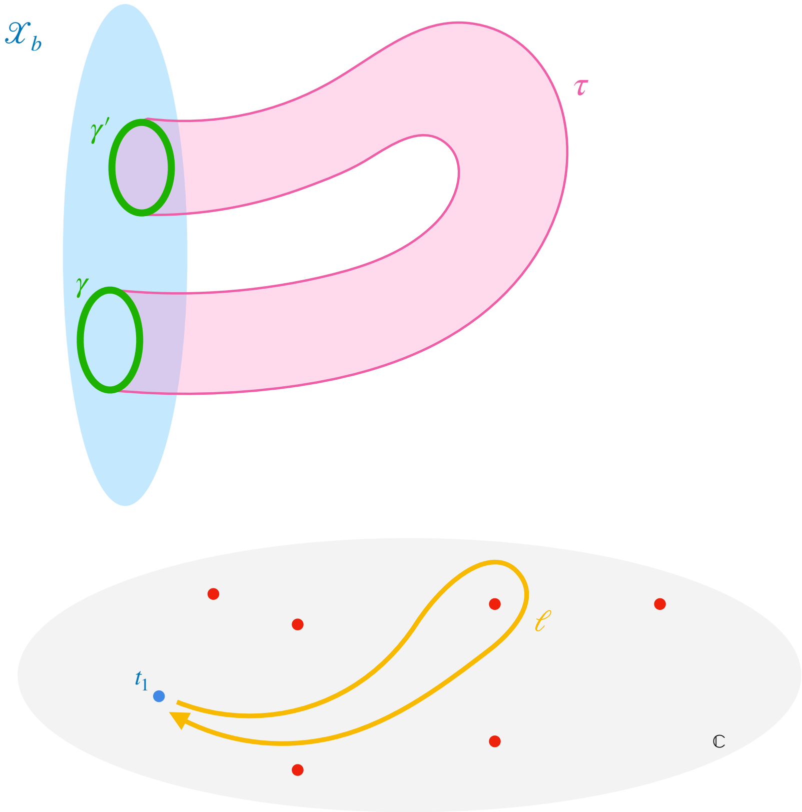

Period of algebraic surface

$$\int_\tau f(x,y)\mathrm{d}x\mathrm{d}y = \int_\ell \left( \int_{\gamma_y} f(x,y)\mathrm{d}x \right) \mathrm{d}y$$

Period of algebraic curve

⟶ Induction on dimension

# Comparison with dimension 1




Thimbles are $n$-cycles obtained by extending $n-1$-cycles along loops.

The monodromy along a loop $\ell$ is an isomorphism of $H_{n-1}(\mathscr{X}_b)$.

If the projection is generic (Lefschetz), singular fibres are simple.
There is a single thimble per critical value.

We get *almost* every possible $n$-cycle by gluing thimbles.

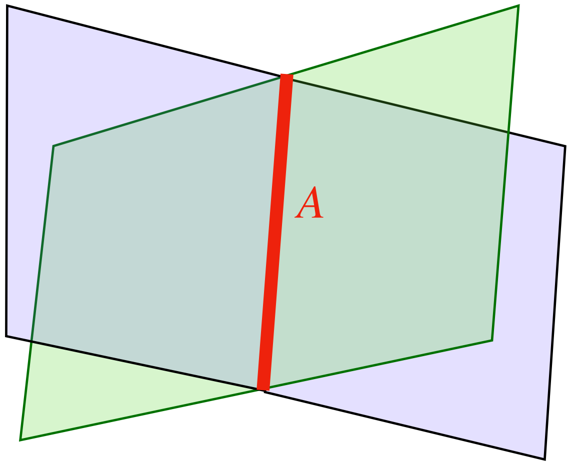
$$H_n(\mathscr{X}_b) \to H_n(\mathscr{X}) \to H_n(\mathscr{X}, \mathscr{X}_b) \to H_{n-1}(\mathscr{X}_b)$$

Possibly nontrivial

*Almost* generated by thimbles

# Obtaining a fibration from a hypersurface

The fibration of $\mathscr{X}$ is given by a hyperplane pencil $\{H_t\}_{t\in\mathbb{P}^1}$, with $\mathscr{X}_t = \mathscr{X} \cap H_t$.




This pencil has an axis $A = \cap_{t\in\mathbb{P}^1} H_t$ that intersects $\mathscr{X}$.

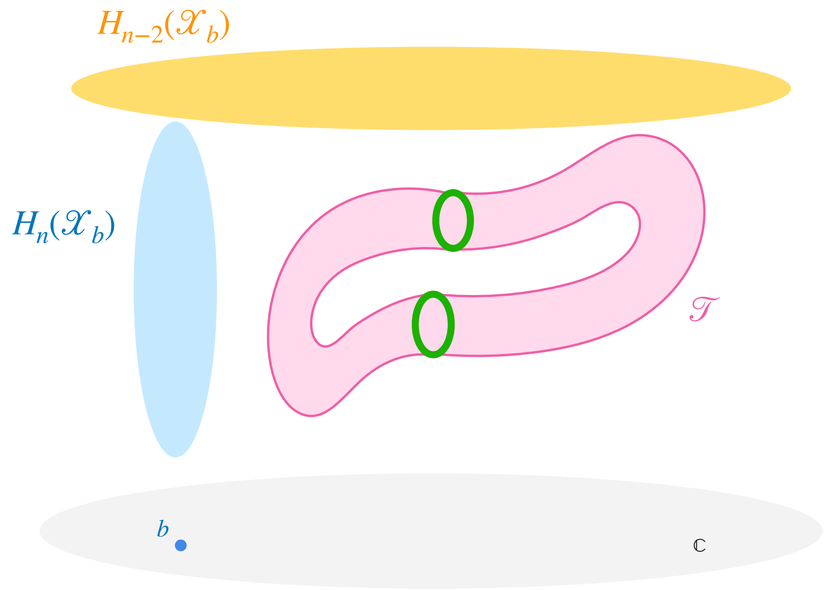
The total space of the fibration is not isomorphic to $\mathscr{X}$, but to a blow up $\mathscr{Y}$ of $\mathscr{X}$ along $\mathscr{X}'$, called the **modification** of $\mathscr{X}$.

We compute $H_n(\mathscr{Y})$, which contains the homology classes of exceptional divisors.
To recover $H_n(\mathscr{X})$ we need to be able to identify these classes.

$$0 \to H_{n-2}(\mathscr{X}') \to H_n(\mathscr{Y}) \to H_n(\mathscr{X}) \to 0$$

# Some complications

Not all cycles of $H_n(\mathscr{Y})$ are lift of loops, and thus not all are combinations of thimbles.




More precisely, we are missing the homology class of the fibre $H_n(\mathscr{X}_b)$
and a section (an extension to $H_{n-2}(\mathscr{X}_b)$ to all of $\mathbb{P}^1$).

We have a filtration $\mathscr{F}^0 \subset \mathscr{F}^1 \subset \mathscr{F}^2 = H_n(\mathscr{Y})$ such that

$$\mathscr{F}^0 \simeq H_n(\mathscr{X}_b)$$
$$\mathscr{F}^1/\mathscr{F}^0 \simeq \mathscr{T}$$
$$\mathscr{F}^2/\mathscr{F}^1 \simeq H_{n-2}(X_b)$$

$\mathscr{T}$ is also known as the **parabolic cohomology** of the local system.

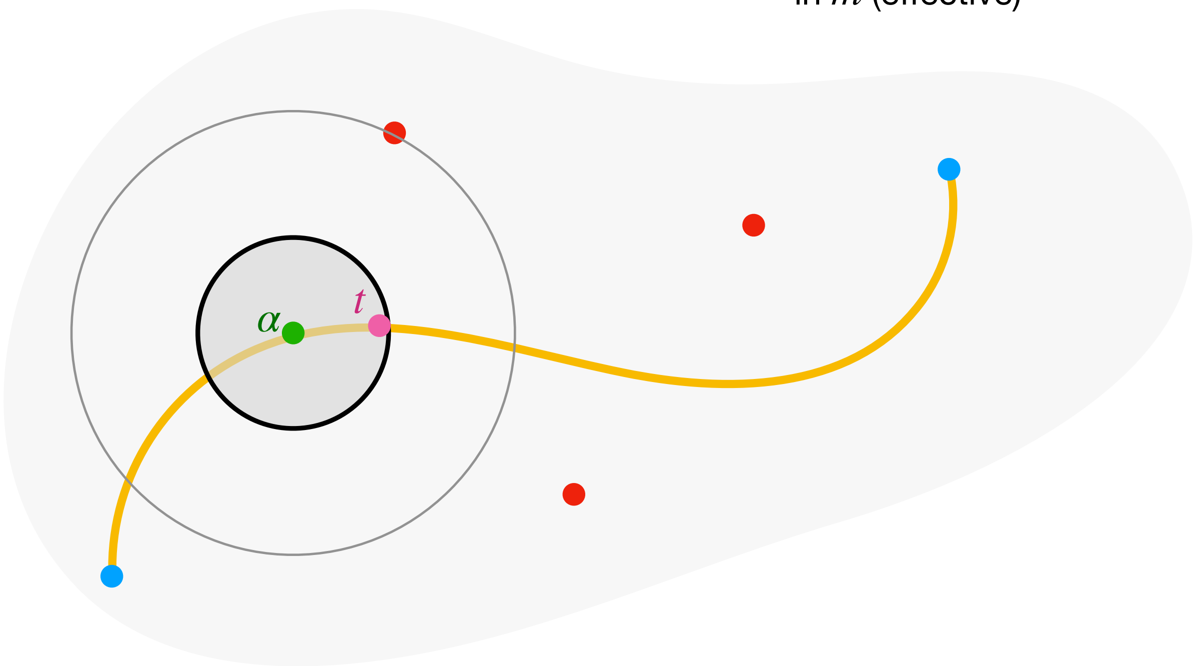
# Monodromy of a differential operator

**[Chudnovsky[2] 90, Van der Hoeven 99, Mezzarobba 2010]**

In a small radius around $\alpha$:

$$\left| f(t) - \sum_{k=0}^{m} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \right| \leq \mathscr{P}(m)2^{-m}$$

polynomial in $m$ (effective)




We compute $f^k(\alpha)$ from $\mathscr{L}$.

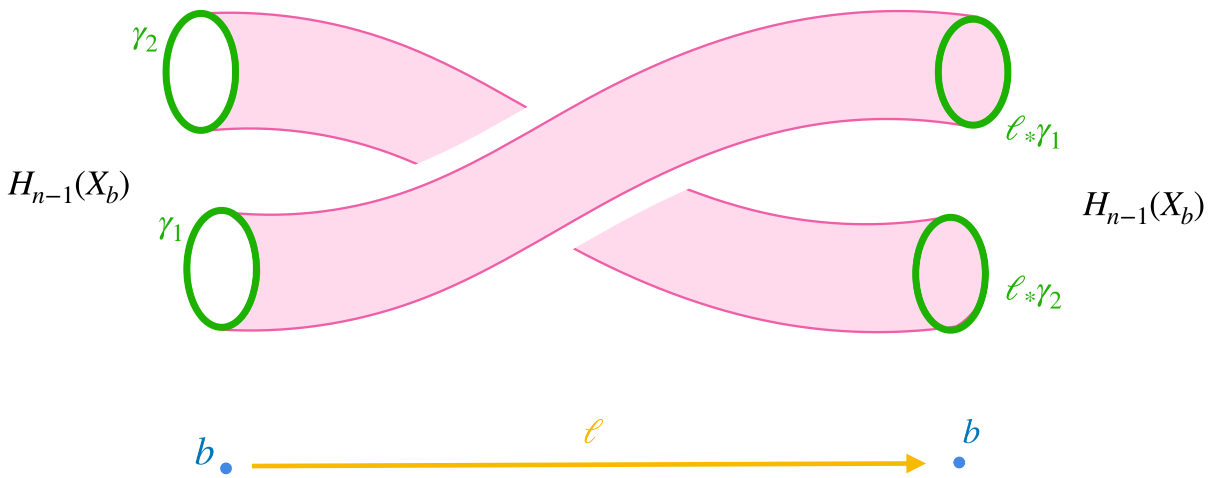
In a disk around $\alpha$, the precision given by the Taylor formula is exponential in its order.

From the derivatives at $\alpha$,
we can recover the derivatives at $t$.

Linear complexity:
Recover $m$ digits in $\mathcal{O}(m)$ operations
(using binary splitting)

# Computing monodromy - I

$$\pi_1(\mathbb{C}\backslash\{\text{critical values}\}) \to GL(H_{n-1}(X_b))$$




Given by periods!

**Tools we use:**

- induction on dimension — we know cycles of $H_{n-1}(X_b)$
- isomorphism between homology and DeRham cohomology → we gain analytical structure
- monodromy of a differential operator (Picard-Fuchs equation) **[Mezzarobba]**

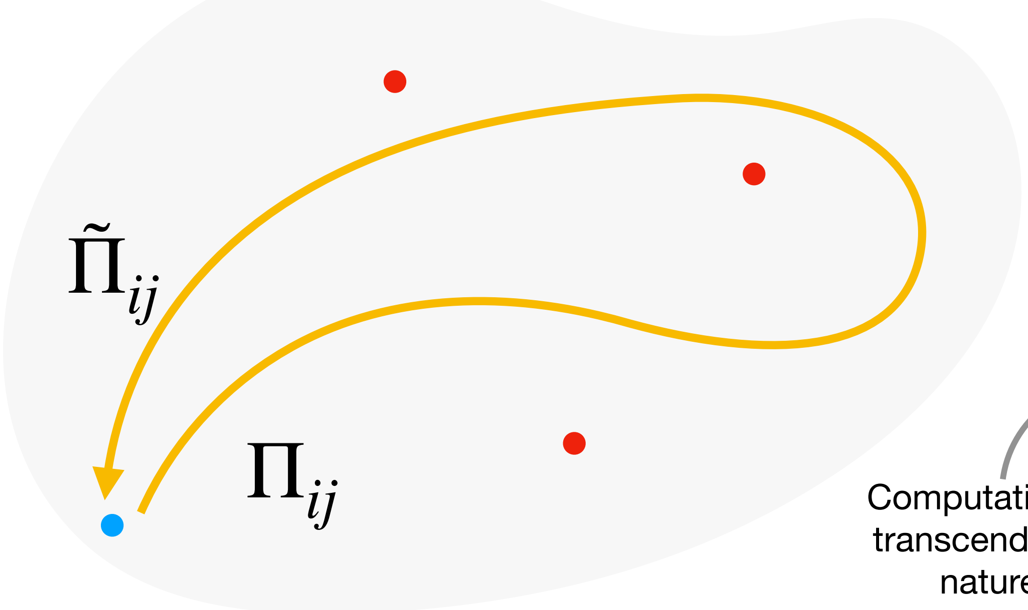
# Computing monodromy - I

$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t \xrightarrow{\text{Analytical continuation}} \tilde{\Pi}_{ij} = \int_{\sum_k c_{kj}\gamma_k} \partial_t^i \omega_t = \sum_k c_{kj} \int_{\gamma_k} \partial_t^i \omega_t$$

Solution to Picard-Fuchs equation of $\omega_t$

$$\tilde{\gamma}_j = \sum_k c_{kj}\gamma_k$$

Globally defined = no monodromy



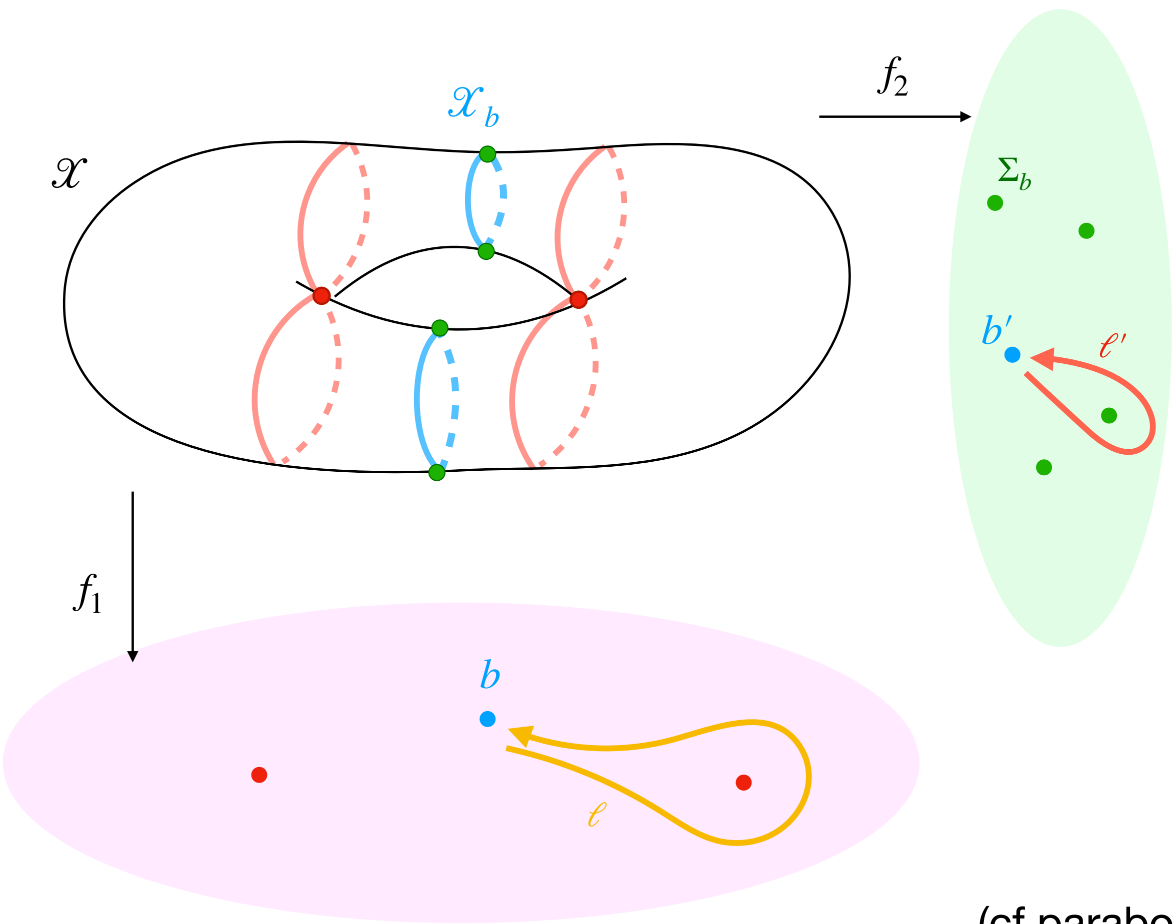

Thus $\tilde{\Pi} = \Pi C$ i.e.

$$\Pi^{-1}\tilde{\Pi} = C \in \mathrm{GL}_r(\mathbb{Z})$$

Computation of transcendental nature

It is sufficient to carry out this computation with precision $< 1/2$ to recover $C$ exactly

# Computing monodromy - II




Critical values of $f_2 : \mathscr{X}_t \to \mathbb{P}^1$ move as $b$ moves in $\mathbb{P}^1$

Thus a loop in $\ell \in \pi_1(\mathbb{P}^1 \backslash \Sigma, b)$ induces a **braid action** on $\pi_1(\mathbb{P}^1 \backslash \Sigma_b, b')$, which lifts to an action on $H_{n-1}(\mathscr{X}_b, \mathscr{X}_{bb'})$.

More precisely, we have that

$$\ell_* \tau_{\ell'}(\gamma) = \tau_{\ell_* \ell'}(\gamma)$$

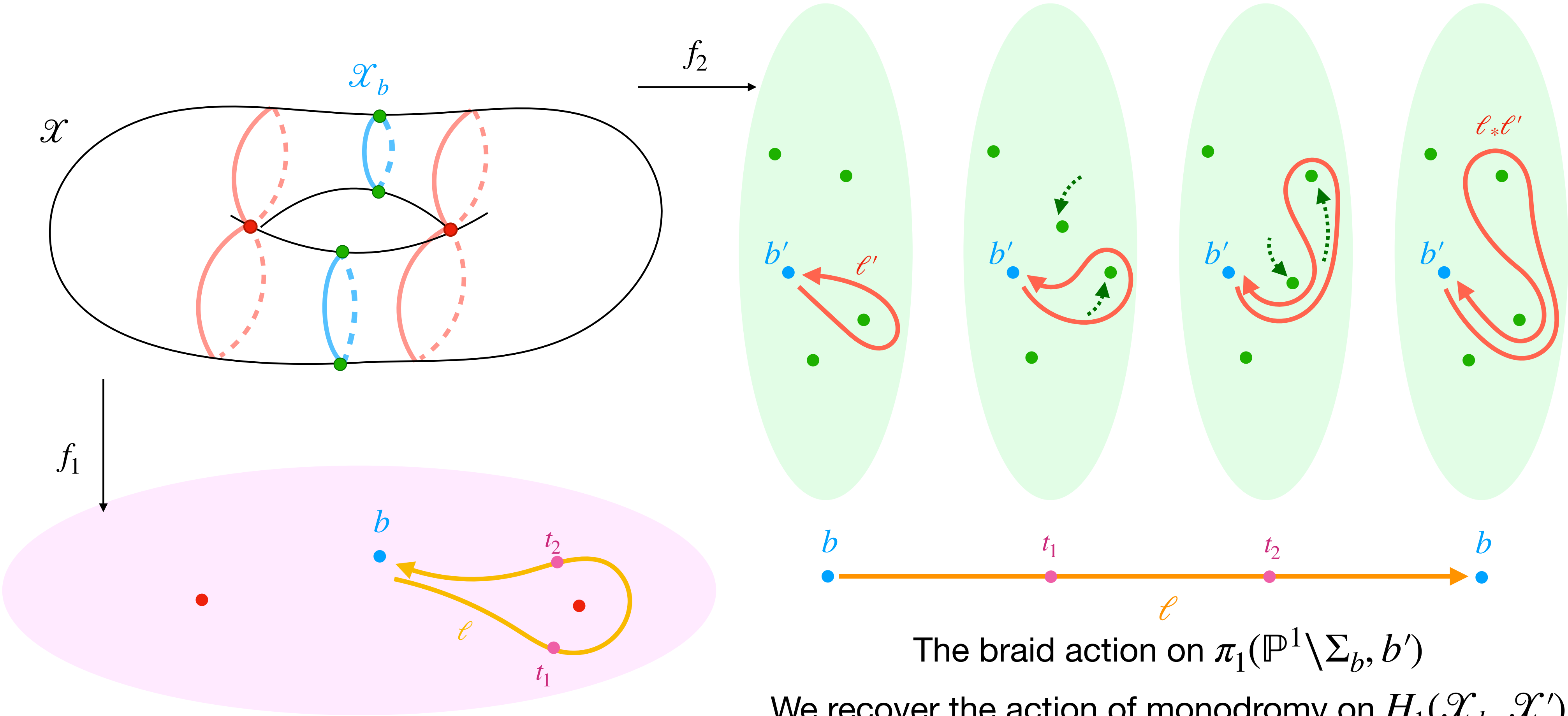
(assuming $\mathscr{X}_{tb'}$ has trivial monodromy with respect to $t$)

Take $b'$ s.t. $\mathscr{X}_{tb'} = \mathscr{X}'$

(cf parabolic cohomology **[Dettweiler, Wewers 2006]**)

# Computing monodromy - II

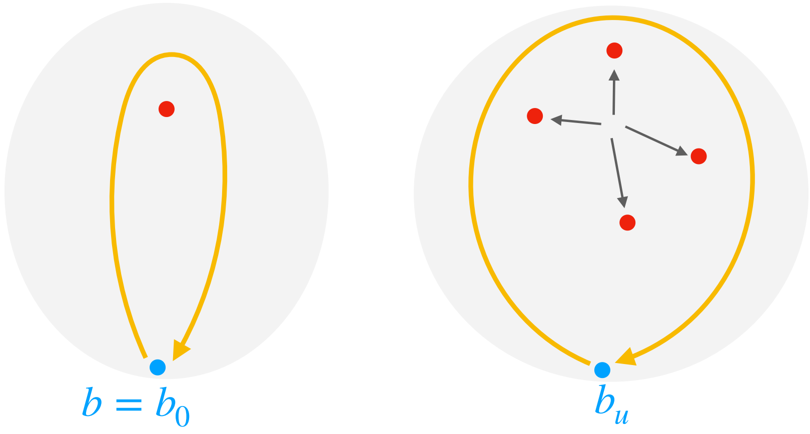
$\ell_* \tau_{\ell'}(\gamma) = \tau_{\ell_* \ell'}(\gamma)$




The braid action on $\pi_1(\mathbb{P}^1 \backslash \Sigma_b, b')$

We recover the action of monodromy on $H_1(\mathscr{X}_b, \mathscr{X}')$

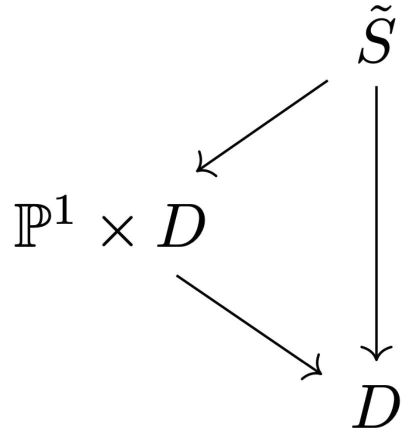
# Morsifications




All $I_1$ fibres

Let $S \to \mathbb{P}^1$ be the fibration of a surface,
with possibly non-Lefschetz fibres.

We consider a **morsification** of $S \to \mathbb{P}^1$
i.e. a family of fibrations $S_u \to \mathbb{P}^1$ parametrised by $u \in D$ (disk)
such that $S_0 \to \mathbb{P}^1$ coincides with $S \to \mathbb{P}^1$,
$S_u \to \mathbb{P}^1$ is a Lefschetz fibration for $u \neq 0$,
and $\tilde{S} \to D$ is a smooth fibration.

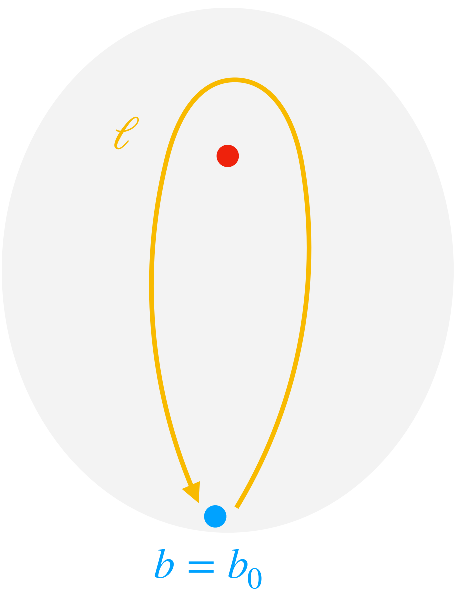


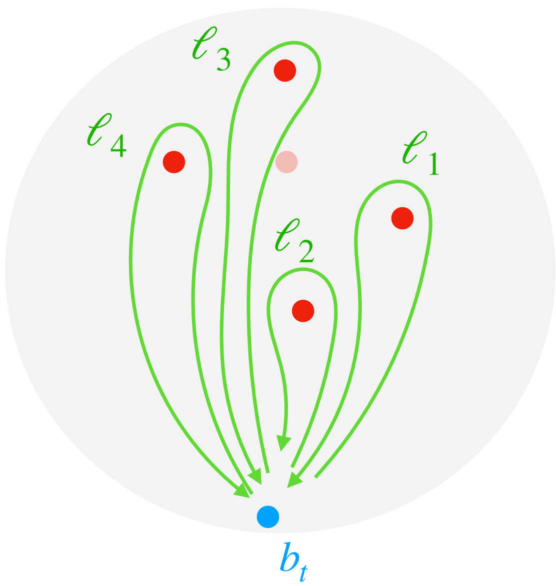

As $H_2(S) = H_2(S_0) \simeq H_2(S_u)$ for $u \neq 0$, we may compute
a description of $H_2(S)$ in terms of thimbles of $S_u$.

Some cycles of $S_u$ can be obtained as extensions in $S$.
These are sufficient to recover the periods of $S$.

# Elliptic surfaces

**Fact:** Morsifications always exist **[Moishezon 1977]**




$$\ell_* = \ell_{4*}\ell_{3*}\ell_{2*}\ell_{1*}$$

**Fact:** The monodromy representation of the morsification is determined by the monodromy representation of $S$. **[Cadavid, Vélez 2009]**

**Kodaira classification**

| | | |
|---|---|---|
| $I_\nu, \nu \geq 1$ | $\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$ | $U^\nu$ |
| $II$ | $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ | $VU$ |
| $III$ | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $VUV$ |
| $IV$ | $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ | $(VU)^2$ |

...

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

**DEMO**

# Results and perspectives

holomorphic periods of quartic surfaces in an hour.

A singular example: Tardigrade family (a very generic family of quartic K3 surfaces).
**[Doran, Harder, PP, Vanhove 2023]**
→ able to embed Néron-Severi lattice in standard K3 lattice

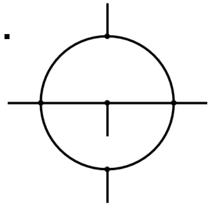


FIGURE 13. The tardigrade graph

Found smooth quartic surface in $\mathbb{P}^3$ with Picard rank 2, 3, 5

$$\mathcal{X} = V\left(\begin{array}{c} X^4 - X^2Y^2 - XY^3 - Y^4 + X^2YZ + XY^2Z + X^2Z^2 - XYZ^2 + XZ^3 \\ -X^3W - X^2YW + XY^2W - Y^3W + Y^2ZW - XZ^2W + YZ^2W - Z^3W + XYW^2 \\ +Y^2W^2 - XZW^2 - XW^3 + YW^3 + ZW^3 + W^4 \end{array}\right)$$

can be applied to more general types of varieties, e.g. complete intersections
up next: K3 surfaces given as double covers of $\mathbb{P}^2$ ramified along sextics.

Bottleneck for accessing higher dimensions is still the order/degree of the differential operators

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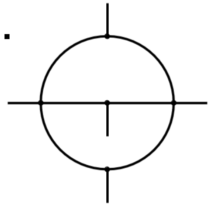

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**Thank you!**